PRODUCT INTEGRALS OF CONTINUOUS RESOLVENTS: EXISTENCE AND NONEXISTENCE

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ABSTRACT

Let $\{[I-\lambda A(t)]^{-1}: 0 \leqq \lambda \leqq \Lambda, 0 \leqq t \leqq T\}$ be a family of resolvents of bounded linear m-dissipative operators $A(t)$ on a Banach space X. Suppose that the map$(\lambda, t, x \mapsto [I-\lambda A(t)]^{-1}x$ is jointly continuous. Then we show it is not necessarily true that for each $x \in X$: (1) the product integral $\lim_{n\to\infty}\Pi_{i=1}^{n}[I-(t/n)A(it/n)]^{-1}x$ exists, (2) the initial value problem $y'(t)=A(t)y(t)$, $y(0)=x$ has a strong solution.

1. Introduction

Consider the evolution equation

$$\frac{dy}{dt} = A(t)y, \quad 0 \leqq t \leqq T, \qquad \text{with given initial value } y(0) \tag{1.1}$$

where for each t, $A(t)$ is an operator on a Banach space $(X, \|\cdot\|)$ and y is X-valued. In the study of this equation, it is customary to impose on A sufficient spatial and temporal hypotheses so as to ensure the existence of a solution y. Spatial hypotheses encompass the notions of continuity (e.g. boundedness, compactness, closedness or Lipschitz continuity of the operator $A(t)$), positivity (e.g. each $A(t)$ is positive, selfadjoint, monotone or accretive) and linearity (e.g. $A(t)$ is a linear, semilinear, quasilinear or nonlinear operator). On the other hand, temporal hypotheses may be categorized according to conditions put directly on $A(\cdot)$ itself or conditions imposed on its resolvent, $J_\lambda(t) = [I-\lambda A(t)]^{-1}$. Conditions on A itself are generally easy to verify, for example, continuity of $A(\cdot)$. Resolvent type conditions, though usually more difficult to check, are useful for applications in the areas of functional and partial differential equations.

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We shall consider the well-known Crandall–Pazy resolvent conditions which suppose that a family of nonlinear operators $A(t)$ on X satisfies for some real number ω,

(i) $\bar{D} = \overline{\mathrm{Dom}(A(t))}$ is independent of t,

(ii) $\mathrm{Dom}(J_\lambda(t)) \supseteq \bar{D}$ for $0 \leqq t \leqq T$ and all $0 < \lambda < \Lambda$, where $\omega\Lambda < 1$,

(iii) each $J_\lambda(t)$ is a Lipschitz mapping with $\langle J_\lambda(t)\rangle_{\mathrm{Lip}} \leqq (1-\lambda\omega)^{-1}$,

(iv) there exist a continuous function $f:[0,T] \to X$ and an increasing function L such that

$$\|J_\lambda(t)x - J_\lambda(s)x\| \leqq \lambda \|f(t) - f(s)\| L(\|x\|)$$
$$\text{for } 0 < \lambda < \Lambda, \quad 0 \leqq s, t \leqq T \quad \text{and} \quad x \in \bar{D}.$$

Crandall and Pazy in [1] prove that these conditions imply the existence of the product integral

$$\text{(1.2)} \qquad \prod_0^t J_{d\xi}(\xi)x \overset{\mathrm{def}}{=} \lim_{n\to\infty} \prod_{i=1}^n J_{t/n}(it/n)x \qquad \text{for all } x \in \bar{D}, \quad t \in [0,T]$$

and that this product integral provides the strong solution $y(t) = \Pi_0^t J_{d\xi}(\xi)y(0)$ to (1.1) whenever such a solution exists. In turn, the solution to the "abstract" ordinary differential equation (1.1) can provide the solution to a concrete partial differential equation. The key is to view (1.1) as a partial differential equation by specially choosing X to be a function space acted on by a particular partial differential operator $A(t)$. This is illustrated in [1] by way of a nonautonomous nonlinear partial differential equation, shown to have a unique solution. Likewise in [5] and [14] general nonautonomous functional differential equations are analyzed and solved by choosing suitable X and A in (1.1) so that conditions (i)–(iv) are satisfied.

Evans in [7], Pierre in [11] and Webb and Badii in [14] all work with variants of the Crandall–Pazy conditions. In particular, each assumes conditions similar or identical to (i), (ii) and (iii) and substitutes a variant of (iv), in order to prove existence of (1.2) or some type of solution to (1.1). We too shall single out condition (iv) for replacement. After all, as shown by the Hille–Yosida theorem, at least in the linear autonomous case, a version of conditions (i), (ii), and (iii) is not only sufficient but also necessary for the existence of (1.2). It is the nonautonomous condition (iv) which seems somewhat unnatural and which we investigate.

An essential feature of (iv) is that the time dependence of $A(t)$ is uniform with respect to the space variable x. Martin [9], Dollard and Friedman [3] and Schechter [12], [13] break away from any assumption of uniform time dependence but assume as a trade-off that A is continuous in its spatial variable. In Sections 2, 3 and 4, we too shall investigate questions of existence of (1.2) when (iv) is replaced by temporal hypotheses not uniform in the space variable.

In Section 4 it is shown that the Crandall–Pazy conditions, while sufficient, are not necessary for the existence of (1.2). Specifically, we exhibit a family $\{A(t)\}_{0 \leqq t \leqq T}$ of bounded linear operators which satisfies conditions (i), (ii) and (iii) but not (iv) or any variant of (iv) and for which the product integral (1.2) exists. In contrast to this example, our investigations in Section 3 will demonstrate that existence of (1.2) does *not* necessarily follow when (iv) is weakened to

(iv)′ $J_\lambda(t)$ is a strongly continuous function of the ordered pair $(\lambda, t) \in [0, \infty) \times [0, T]$,

even if (i), (ii) and (iii) are replaced by the simpler and stronger hypotheses

(i)′ each $A(t)$ is a bounded linear operator with $\mathrm{Dom}(A(t)) \equiv X$,

(ii)′ $\mathrm{Dom}(J_\lambda(t)) \equiv X$, and

(iii)′ $\langle J_\lambda(t)\rangle_{\mathrm{Lip}} \leqq 1$ and all t and λ.

To begin, we first examine a case of existence of the product integral.

2. Product integrals of Lipschitz continuous resolvents

Given a Banach space $(X, \|\cdot\|)$, let $\mathrm{Lip}(X)$ denote the set of Lipschitz operators T defined on all of X. Thus for each $T \in \mathrm{Lip}(X)$, the Lipschitz seminorm of T:

$$\langle T\rangle_{\mathrm{Lip}} = \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\|Tx - Ty\|}{\|x - y\|},$$

is finite. We shall consider $\mathrm{Lip}(X)$ as a Banach space, which is the case under the Lipschitz norm given by

$$\|T\|_{\mathrm{Lip}} = \|T(0)\| + \langle T\rangle_{\mathrm{Lip}}.$$

THEOREM 2.1. *Let $\{A(t)\}_{0 \leqq t \leqq t}$ be a family of Lipschitz operators on X satisfying: $t \mapsto A(t)x$ is continuous on $0 \leqq t \leqq T$ for every $x \in X$, and $\langle A(t)\rangle_{\mathrm{Lip}} \leqq M$ for $0 \leqq t \leqq T$. Then,*

(i) *the initial value problem* (1.1) *has a unique classical solution $y(t)$,*

(ii) *the product integral $\Pi_0^t J_{d\xi}(\xi)x$ as defined by* (1.2) *exists for every $x \in X$, $0 \leqq t \leqq T$ and it equals $y(t)$.*

PROOF. Clearly, continuity in t for each fixed x and the Lipschitz condition imply that A is jointly continuous on the strip $[0,T]\times X$. Hence A satisfies the hypotheses of the Picard existence theorem for ordinary differential equations (cf. [8, p. 322]), and this theorem yields the solution we seek in (i).

As for the proof of (ii), given $t\in(0,T]$, let $\{t_i\}_{i=0}^n$ be the partition of $[0,t]$ with $t_i = it/n$, $i=0,1,\cdots,n$. Now if $x\in X$ and y is the solution of (1.1) with the initial condition $y(0)=x$, then y satisfies

$$y(t_i)=y(t_{i-1})+\int_{t_{i-1}}^{t_i} A(s)y(s)\,ds, \qquad i=1,\cdots,n. \tag{2.3}$$

Letting $P_i = I-(t_i-t_{i-1})A(t_i)$ and $y_i = y(t_i)$, (2.3) can be written as

$$P_iy_i = y_{i-1}+\delta_i \qquad \text{where } \delta_i=\int_{t_{i-1}}^{t_i} A(s)y(s)\,ds-(t_i-t_{i-1})A(t_i)y(t_i). \tag{2.4}$$

Next define the sequence of vectors $z_0 = x$, and for $i\geqq 1: z_i = P_i^{-1}z_{i-1}$, where we assume that n is large enough so that $\|(t_i-t_{i-1})A(t_i)\|_{\mathrm{Lip}}<1$. Therefore, if $\alpha_i=\|y_i-z_i\|$ and $\mu_i=\max\{1,\langle P_i^{-1}\rangle_{\mathrm{Lip}}\}$ then from (2.4):

$$\begin{aligned}\alpha_i &= \|P_i^{-1}(P_iy_i)-P_i^{-1}(P_iz_i)\|\\ &\leqq \langle P_i^{-1}\rangle_{\mathrm{Lip}}\|y_{i-1}-z_{i-1}+\delta_i\|\\ &\leqq \mu_i(\alpha_{i-1}+\|\delta_i\|).\end{aligned} \tag{2.5}$$

Now, for each i, multiply both sides of (2.5) by $\Pi_{j=i+1}^n\mu_j$, and then sum over i for $1\leqq i\leqq n$. Since $\alpha_0=0$, it follows that for $n\geqq 2tM$,

$$\alpha_n\leqq\sum_{i=1}^n\left(\prod_{j=i}^n\mu_j\right)\|\delta_i\|\leqq\left(\prod_{j=1}^n\mu_i\right)\sum_{i=1}^n\|\delta_i\|\leqq e^{2tM}\sum_{i=1}^n\|\delta_i\|,$$

where we have used the bounds $\mu_j\leqq 1/(1-tM/n)\leqq 1+2tM/n<e^{2tM/n}$.

For each i we have

$$\begin{aligned}\|\delta_i\| &= \left\|\int_{t_{i-1}}^{t_i}(y'(s)-y'(t_i))\,ds\right\|\\ &\leqq (t_i-t_{i-1})\sup\left\{\|y'(s)-y'(t_i)\|: s\in\left[\frac{(i-1)t}{n},\frac{it}{n}\right]\right\}.\end{aligned}$$

Since $y'(s)$ is continuous, given $\varepsilon>0$ there exists n_0 such that for all $n\geqq n_0$, $\|\delta_i\|\leqq\varepsilon e^{-2tM}(t_i-t_{i-1})/t$. Hence, for all such n,

$$\left\|y(t)-\prod_{i=1}^n\left[I-\frac{t}{n}A\left(\frac{it}{n}\right)\right]^{-1}x\right\|=\alpha_n\leqq e^{2tM}\sum_{i=1}^n\varepsilon e^{-2tM}(t_i-t_{i-1})/t=\varepsilon.$$

Therefore, $\Pi_0^t J_{d\xi}(\xi)x$ exists and equals $y(t)$ for all $t \in [0,T]$.

Throughout the remainder of this section, assume that $A:[0,T]\times X \to X$ is an arbitrary nonlinear operator with resolvent $J_\lambda(t)$ which satisfies:

(2.1) $J_\lambda(t) \in \mathrm{Lip}(X)$ for each (λ, t) in the rectangle $[0,\Lambda]\times[0,T]$,

(2.2) $(\lambda,t) \mapsto J_\lambda(t)$ is a continuous map from $[0,\Lambda]\times[0,T]$ into $\mathrm{Lip}(X)$.

LEMMA 2.1. *For each $t\in[0,T]$, $A(t)$ is a Lipschitz operator on X, and $M \equiv \sup_{t\in[0,T]}\langle A(t)\rangle_{\mathrm{Lip}}$ is finite.*

PROOF. Since $J_0(t) = I$ and J is continuous, there exists $\mu > 0$ such that $\langle I - J_\mu(t)\rangle_{\mathrm{Lip}} < 0.5$ for all $t \in [0,T]$. It follows (cf. [10, p. 66]) that for each t, $J_\mu^{-1}(t)$ exists in $\mathrm{Lip}(X)$ and satisfies $\langle J_\mu^{-1}(t)\rangle_{\mathrm{Lip}} \leqq 1/(1-\langle I - J_\mu(t)\rangle_{\mathrm{Lip}})$. Therefore,

$$\langle A(t)\rangle_{\mathrm{Lip}} \leqq \frac{1}{\mu}\left\{1 + \frac{1}{1-\langle I - J_\mu(t)\rangle_{\mathrm{Lip}}}\right\} \leqq \frac{3}{\mu}. \qquad \blacksquare$$

LEMMA 2.2. *If $\{t_n\}_{n=1}^\infty \subseteq [0,T]$ and $t_n \to t_0$, then, for all $x \in X$, $A(t_n)x \to A(t_0)x$.*

PROOF. Let μ be such that $\langle I - J_\mu(t_0)\rangle_{\mathrm{Lip}} < 1$. Hence $J_\mu^{-1}(t_0) \in \mathrm{Lip}(X)$. The proof is completed by identifying $J_\mu(t_0)$ with F and $J_\mu(t_0) - J_\mu(t_n)$ with G_n in Lemma 2.3 below.

LEMMA 2.3. *Suppose $F \in \mathrm{Lip}(X)$ has inverse F^{-1} in* Lip(*X*) *and $\{G_n\}_{n=1}^\infty$ is a sequence in* Lip(*X*) *with $\|G_n\|_{\mathrm{Lip}} \to 0$ as $n\to\infty$. Then eventually $F - G_n$ has inverse in* Lip(*X*) *and $\lim_{n\to\infty}(F-G_n)^{-1}(x) = F^{-1}(x)$ for each $x \in X$.*

PROOF. Eventually $\|G_n\|_{\mathrm{Lip}} < \|F^{-1}\|_{\mathrm{Lip}}^{-1}$, which implies that $(F-G_n)^{-1}$ exists in $\mathrm{Lip}(X)$ [10, p. 66]. Taking this to be the case, given $x \in X$ let $z_n = (F-G_n)^{-1}(x)$ and $z = F^{-1}(x)$. Then

$$\begin{aligned}\|z_n - z\| &\leqq \|F^{-1}\|_{\mathrm{Lip}}\|F(z_n) - F(z)\| = \|F^{-1}\|_{\mathrm{Lip}}\|G_n(z_n)\| \\ &\leqq \|F^{-1}\|_{\mathrm{Lip}}(\|G_n(z_n) - G_n(z)\| + \|G_n(z) - G_n(0)\| + \|G_n(0)\|) \\ &\leqq \|F^{-1}\|_{\mathrm{Lip}}\|G_n\|_{\mathrm{Lip}}(\|z_n - z\| + \|z\| + 1),\end{aligned}$$

which implies that

$$\|z_n - z\| \leqq \frac{\|F^{-1}\|_{\mathrm{Lip}}\|G_n\|_{\mathrm{Lip}}(\|z\|+1)}{1 - \|F^{-1}\|_{\mathrm{Lip}}\|G_n\|_{\mathrm{Lip}}}. \qquad \blacksquare$$

Applying Lemmas 2.1 and 2.2 to Theorem 2.1 now yields

THEOREM 2.2. *If A has resolvent which satisfies conditions* (2.1) *and* (2.2), *the product integral $\Pi_0^t J_{d\xi}(\xi)x$ exists for every $x \in X$ and $t \in [0, T]$.*

REMARKS. (1) Though $A(t)$ is not assumed to be dissipative, contained in condition (2.2) is the strong assumption that for each fixed t, $\lim_{\lambda \downarrow 0} \|J_\lambda(t) - I\|_{\mathrm{Lip}} = 0$. While this assumption may limit the scope of Theorem 2.2, we have included this theorem mainly to put into sharper focus the nonexistence results presented in Section 3 where Lipschitz continuity is weakened to strong continuity.

(2) If each $A(t)$ is additionally assumed to be a linear operator on X, it can be shown that the product integral $\Pi_0^t J_{d\xi}(\xi)$ converges in the operator-norm topology for every t in $[0, T]$.

3. Counterexample

We shall exhibit a family of operators $\{A(t)\}_{0 \le t \le 1}$ on a Banach space X satisfying

(3.1) for each $t \in [0,1]$, $A(t)$ is a bounded linear operator defined on all of X,

(3.2) for each $t \in [0,1]$, $A(t)$ is m-dissipative (i.e. conditions (ii)′ and (iii)′ of Section 1 are satisfied),

(3.3) $J_\lambda(t)x = [I - \lambda A(t)]^{-1}x$ is a jointly continuous function of the triple $(\lambda, t, x) \in [0, \infty) \times [0,1] \times X$,

and such that:

PROPOSITION 3.1. *There is a $w \in X$ such that $\Pi_0^t J_{d\xi}(\xi)w$ does not exist for any $t \in (0,1]$.*

PROPOSITION 3.2. *There is a $y(0) \in X$ such that the initial value problem* (1.1) *has no strong solution for any $T > 0$.*

As a consequence of these propositions, it is clear that the following two statements:

for every $x_0 \in X$, there is at least one $t \in (0,1]$ for which the product integral $\Pi_0^t J_{d\xi}(\xi)x_0$ exists,

for every $y(0) \in X$, there exists $T > 0$ such that the initial value problem (1.1) *has a strong solution $y(t)$,*

are false.

By a strong solution to the initial value problem (1.1) with initial value x, is meant a function $y:[0,T]\to X$ such that

(1) y is continuous on $[0,T]$ and $y(0)=x$,

(2) y is absolutely continuous on compact subsets of $(0,T)$,

(3) y is differentiable a.e. on $(0,T)$ and satisfies (1.1) a.e.

The continuity condition (3.3) is equivalent to separate continuity in λ, t and x of the map $(\lambda,t,x)\mapsto J_\lambda(t)x$ when A satisfies (3.1), (3.2) and $\|A(t)\|$, the operator norm of $A(t)$, is bounded. However, this case is of no interest with respect to the preceding propositions, for it implies that for fixed x and t, each $A(t)x$ is the uniform limit as $n\to\infty$ of the t-continuous Yosida approximants $n(J_{1/n}(t)x-x)$. Hence $A(t)$ is strongly continuous, so, as follows from Theorem 2.1, the product integral (1.2) exists. In general, though, for A with $\|A(t)\|$ unbounded on $[0,1]$, condition (3.3) is stronger than separate continuity.

In the construction of the counterexample, we first define the Ω-norm of a sequence $x=(x_1,x_2,\cdots)$ of elements drawn from ℓ_∞ as

$$|x|_\Omega=\sup\left\{\left\|\sum_{i=1}^{N}\theta_i x_i\right\|_\infty : N<\infty \text{ and } \theta_1,\theta_2,\cdots\in\{0,1\}\right\}.$$

Our counterexample will be set in the space

$$\Omega \overset{\text{def}}{=} \left\{(x_i)_{i=1}^\infty : x_i\in\ell_\infty, \lim_{i\to\infty}\|x_i\|_\infty=0 \text{ and } |(x_i)_i|_\Omega<\infty\right\}.$$

LEMMA 3.1. *$(\Omega,|\cdot|_\Omega)$ is a Banach space.*

PROOF. Clearly $|\cdot|_\Omega$ is a norm on the space of sequences of elements drawn from ℓ_∞ Now consider the space $Z=\{(x_i)_{i=1}^\infty : x_i\in\ell_\infty \text{ and } |(x_i)_i|_\Omega<\infty\}$. The proof that Z is complete under the Ω-norm follows the same steps as the ordinary proof of the completeness of ℓ_∞. But Ω is a closed subspace of Z, as is clear from the inequality

(3.4) $\|x_k\|_\infty\leqq|x|_\Omega$ for every index k applied to any $x=(x_i)_{i=1}^\infty$ in Ω.

Therefore, Ω is itself a complete normed linear space.

The next lemma shows that Ω contains a vector having components which might usually be associated with a divergent series.

LEMMA 3.2. *There exists $w=(w_1,w_2,w_3,\cdots)\in\Omega$ which satisfies*

(3.5) $\left\|\sum_{j=1}^{M}w_j\right\|_\infty=1$ *and* $\left\|\sum_{j=1}^{N}w_j\right\|_\infty=\frac{1}{2}$, *for infinitely many choices of M and N.*

PROOF. Let $e_k = (0,\cdots,0,1,0,\cdots)$ be the vector with 1 in the kth component and 0's elsewhere. Let $w_1 = e_1$. If n and j are positive integers with $2^{n-1} < j \leqq 2^n$, we define

$$w_j = \frac{(-1)^n}{2^{n-1}}(e_n + e_{n+1}). \tag{3.6}$$

Hence

$$\sum_{j=2^{n-1}+1}^{2^n} w_j = \left(\frac{2^n - 2^{n-1}}{2^{n-1}}\right)(-1)^n(e_n + e_{n+1}) = (-1)^n(e_n + e_{n+1}),$$

and

$$\sum_{j=2^{n-1}+1}^{(3/4)2^n} w_j = \frac{1}{2}\sum_{j=2^{n-1}+1}^{2^n} w_j = \frac{(-1)^n}{2}(e_n + e_{n+1}).$$

Therefore,

$$\sum_{j=1}^{2^n} w_j = w_1 + \sum_{k=1}^{n}\sum_{j=2^{k-1}+1}^{2^k} w_j = e_1 - (e_1 + e_2) + (e_2 + e_3) - \cdots + (-1)^n(e_n + e_{n+1})$$
$$= (-1)^n e_{n+1},$$

and

$$\sum_{j=1}^{(3/4)2^n} w_j = \sum_{j=1}^{2^{n-1}} w_j + \frac{(-1)^n}{2}(e_n + e_{n+1}) = \frac{(-1)^n}{2}(e_{n+1} - e_n).$$

Thus

$$\left\|\sum_{j=1}^{2^n} w_j\right\|_\infty = 1, \qquad \left\|\sum_{j=1}^{(3/4)2^n} w_j\right\|_\infty = \frac{1}{2}. \qquad \blacksquare$$

DEFINITIONS 3.1. (1) Let $\mathcal{T}(t)$ denote the function whose graph is the triangular spike with vertices at $(-1,0)$, $(0,1)$ and $(1,0)$. Thus

$$\mathcal{T}(t) = \begin{cases} 0, & \text{if } |t| \geqq 1; \\ 1 - |t|, & \text{if } |t| \leqq 1. \end{cases}$$

(2) For each $p = 1,2,\cdots$, let $a_p(t) = -2^{p+2}\mathcal{T}(2^{p+2}t - 3)$.

(3) For each $t \in (0,1]$ and $x \in \Omega$ define $A(t)$ to be the diagonal operator given by $A(t)x = (a_p(t)x_p)_{p=1}^\infty$, and let $A(0) = 0$.

REMARKS 3.1. (1) We see that each a_p is a downward triangular spike, and the a_p's have nonoverlapping supports. Hence, given $x = (x_p)_{p=1}^\infty$,

$$A(t)x = (0,\cdot \quad 0, a_p(t)x_p, 0, \cdots), \qquad \text{when } t \in [2^{-(p+1)}, 2^{-p}].$$

(2) Keep in mind that x_p and $a_p(t)x_p$ are themselves elements of ℓ_∞, not scalars.

(3) Observe that each a_p satisfies: if $0 < t \leqq 1$ and $p \geqq \log_2(1/t)$ then

$$\int_0^t a_p(s)\,ds = -1 \quad \text{and} \quad a_p(\sigma) = 0 \qquad \text{for all } t \leqq \sigma \leqq 1.$$

(4) The idea of constructing a function, like A, as a sequence of projection operators of increasing norms appears in a similar construction in Dieudonné's paper [2].

LEMMA 3.3. *The operators* $\{A(t)\}_{0 \leqq t \leqq 1}$ *are bounded, linear m-dissipative on* Ω *and satisfy* (3.3).

PROOF. Remark 1 above shows that each $A(t)$ is bounded, linear, and that for each $x \in \Omega$,

$$J_\lambda(t)x = (x_1, x_2, \cdots, (1 - \lambda a_p(t))^{-1} x_p, x_{p+1}, \cdots) \qquad \text{when } t \in [2^{-(p+1)}, 2^{-p}].$$

To prove m-dissipativeness, we will need the inequality: For any $\theta \in [0,1]$ and $z_1, z_2 \in \ell_\infty$,

$$\begin{aligned} \| z_1 + \theta z_2 \|_\infty &= \| (1-\theta) z_1 + \theta (z_1 + z_2) \|_\infty \leqq (1-\theta) \| z_1 \|_\infty + \theta \| z_1 + z_2 \|_\infty \\ &\leqq \max(\| z_1 \|_\infty, \| z_1 + z_2 \|_\infty). \end{aligned}$$

Hence, if $\lambda > 0$, $t \in [2^{-(p+1)}, 2^{-p}]$ and $x \in \Omega$, then for any finite set $F \subseteq \{1,2,3,\cdots\}$ with $p \notin F$, we have

$$\max\left\{ \left\| \sum_{i \in F} x_i \right\|_\infty, \left\| \sum_{i \in F} x_i + \theta x_p \right\|_\infty \right\} \leqq \max\left\{ \left\| \sum_{i \in F} x_i \right\|_\infty, \left\| \sum_{i \in F} x_i + x_p \right\|_\infty \right\} \leqq | x |_\Omega .$$

Therefore, for $\theta = 1/(1 - \lambda a_p(t))$,

$$\begin{aligned} | J_\lambda(t) x |_\Omega &= \sup\left\{ \max\left\{ \left\| \sum_{i \in F} x_i \right\|_\infty, \left\| \sum_{i \in F} x_i + \theta x_p \right\|_\infty \right\} : F \subseteq \{1,2,\cdots\} \text{ is finite and } p \notin F \right\} \\ &\leqq | x |_\Omega . \end{aligned}$$

Finally, we verify (3.3). In the case $0 < t \leqq 1$, (3.3) follows easily from the continuity of $A(t)$. We leave the details to the reader. For $t = 0$, let $\{t_p\}$ be any sequence which approaches zero and such that $t_p \in [2^{-(p+1)}, 2^{-p}]$, $p = 1,2,\cdots$. Then for x, $y \in \Omega$ and λ, $\mu \geqq 0$, we have $J_\lambda(0)x = x$ and

$$\begin{aligned} | J_\lambda(0)x - J_\mu(t_p) y |_\Omega &\leqq | x - y |_\Omega + | J_\mu(t_p) y - y |_\Omega \\ &= | x - y |_\Omega + \| ([1 - \mu a_p(t_p)]^{-1} - 1) y_p \|_\infty \\ &\leqq | x - y |_\Omega + \| y_p \|_\infty , \end{aligned}$$

which approaches 0 as (μ, t_p, y) approaches $(\lambda, 0, x)$. ■

We shall require one more preliminary lemma in order to complete the counterexample.

LEMMA 3.4. *Given* $x \in \Omega$, *if* $\Pi_0^t J_\xi(\xi)x$ *exists, it must equal* $(\exp(\int_0^t a_p(s)dx)x_p)_{p=1}^\infty$.

PROOF. Let $y_p = \exp(\int_0^t a_p(s)ds)x_p$. Then, since a_p is continuous, it easily follows (cf. [4, p. 52]) that $y_p = \lim_{n\to\infty}\Pi_0^n[1-(t/n)a_p(it/n)]^{-1}x_p$.

Suppose $\Pi_0^t J_{d\xi}(\xi)x$ equals the vector $(z_p)_{p=1}^\infty$, where for each p, $z_p = \{z_p^{(j)}\}_{j=1}^\infty$ and $x_p = \{x_p^{(j)}\}_{j=1}^\infty$. Then for each pair of indices $\mathcal{P}$ and $\mathcal{J}$, inequality (3.4) yields:

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left|\prod_{i=1}^n J_{t/n}(it/n)x - (z_p)_{p=1}^\infty\right|_\Omega \\
&= \lim_{n\to\infty}\left|\left(\prod_{i=1}^n[1-(t/n)a_p(it/n)]^{-1}x_p - z_p\right)_{p=1}^\infty\right|_\Omega \\
&\geqq \limsup_{n\to\infty}\sup_{p\geqq 1}\sup_{j\geqq 1}\left|\prod_{i=1}^n[1-(t/n)a_p(it/n)]^{-1}x_p^{(j)} - z_p^{(j)}\right| \\
&\geqq \left|\lim_{n\to\infty}\prod_{i=1}^n[1-(t/n)a_{\mathcal{P}}(it/n)]^{-1}x_{\mathcal{P}}^{(\mathcal{J})} - z_{\mathcal{P}}^{(\mathcal{J})}\right| = |y_{\mathcal{P}}^{(\mathcal{J})} - z_{\mathcal{P}}^{(\mathcal{J})}| \geqq 0.
\end{aligned}$$

Therefore, $y_p = z_p$ for every index p, and the proof is complete.

PROOF OF PROPOSITION 3.1. With w taken from Lemma 3.2 and given $t \in (0,1]$, consider

$$\begin{aligned}
\Delta_q(t) &\overset{\text{def}}{=} \left|\prod_{i=1}^q[I-(t/q)A(it/q)]^{-1}w - \left(\exp\left(\int_0^t a_p(s)ds\right)w_p\right)_{p=1}^\infty\right|_\Omega \\
&= |(g(p,q)w_p)_{p=1}^\infty|_\Omega,
\end{aligned}$$

where we define $g(p,q) = \Pi_{i=1}^q[1-(t/q)a_p(it/q)]^{-1} - \exp(\int_0^t a_p(s)ds)$. By the previous lemma, it will suffice to show that $\Delta_q(t)$ is bounded away from zero for all q. Remarks 3.1 give that

$$g(p,q) = 1 - 1/e \qquad \text{for all integers } p \text{ and } q \text{ such that } p \geqq \log_2(q/t).$$

Therefore, for each positive integer q, Lemma 3.2 shows there exist integers $M > N > \log_2(q/t)$ such that

$$\Delta_q(t) \geqq \left\|\sum_{p=N}^M g(p,q)w_p\right\|_\infty \geqq \left(1-\frac{1}{e}\right)\left|\left\|\sum_{p=1}^M w_p\right\|_\infty - \left\|\sum_{p=1}^N w_p\right\|_\infty\right| = \frac{1}{2}\left(1-\frac{1}{e}\right).$$

The proof is finished.

There is a more general definition of the product integral $\prod_0^t J_{d\xi}(\xi)w$: For each $\varepsilon > 0$ there exists a partition τ of $[0,t]$ such that if $\{\sigma_i\}_{i=1}^m$ is any partition of $[0,t]$ which includes τ (i.e. σ is a refinement of τ) then

$$\left| \prod_0^t J_{d\xi}(\xi)w - \prod_{i=1}^m J_{\sigma_i - \sigma_{i-1}}(\sigma_i)w \right|_\Omega \leqq \varepsilon .$$

This leads to

COROLLARY 3.1. *With A and w as in Proposition 3.1, the product integral $\prod_0^t J_{d\xi}(\xi)w$ does not exist even if convergence of the product integral is taken in the sense of successive refinements of partitions described above.*

PROOF. It will suffice to show that given $0 < t \leqq 1$, if τ is a partition of the interval $[0,t]$, there is a refinement $\sigma = \{\sigma_i\}_{i=0}^m$ of τ such that

$$\Delta_\sigma(t) \stackrel{\text{def}}{=} \left| \left(\left\{ \prod_{i=1}^m [1 - (\sigma_i - \sigma_{i-1})a_p(\sigma_i)]^{-1} - \exp\left(\int_0^t a_p(s)ds \right) \right\} w_p \right)_{p=1}^\infty \right|_\Omega$$
$$\stackrel{\text{def}}{=} |(g(p,\sigma)w_p)_{p=1}^\infty|_\Omega \geqq \tfrac{1}{2}(1 - 1/e).$$

In fact, let σ equal τ itself, where τ is given by: $0 = \tau_0 < \tau_1 < \cdots < \tau_m = t$. Now set $\mathcal{P}$ equal to a positive integer which satisfies $2^{-\mathcal{P}} \leqq \tau_1$. Then, by Lemma 3.2 there exist integers $M > N > \mathcal{P}$ such that

$$\Delta_\tau(t) \geqq \left\| \sum_{p=N}^M g(p,\tau)w_p \right\|_\infty = \left(1 - \frac{1}{e}\right) \left\| \sum_{p=N}^M w_p \right\|_\infty = \frac{1}{2}\left(1 - \frac{1}{e}\right). \qquad \blacksquare$$

PROOF OF PROPOSITION 3.2. Let A be as in Definitions 3.1. We first show that for $X = \Omega$, if $y(t)$ is a strong solution to (1.1) and $y(0) = (x_p)_{p=1}^\infty$, then

$$y(t) = \left(\exp\left(\int_0^t a_p(s)ds \right) x_p \right)_{p=1}^\infty \qquad \text{for all } t \in [0,1]. \tag{3.7}$$

On $[0,1]$, suppose $y(t) = (y_p(t))_{p=1}^\infty$, where for each p, $y_p(t) = \{y_p^{(j)}(t)\}_{j=1}^\infty$. Then for each pair of indices $\mathcal{P}$ and $\mathcal{J}$:

$$0 = \lim_{h \to 0} \left| \left(\frac{y_p(t+h) - y_p(t)}{h} - a_p(t)y_p(t) \right)_{p=1}^\infty \right|_\Omega$$
$$\geqq \lim_{h\to 0} \sup_{p \geqq 1} \sup_{j \geqq 1} \left\| \frac{y_p^{(j)}(t+h) - y_p^{(j)}(t)}{h} - a_p(t)y_p^{(j)} \right\|_\infty$$
$$\geqq \lim_{h \to 0} \frac{y_{\mathcal{P}}^{(\mathcal{J})}(t+h) - y_{\mathcal{P}}^{(\mathcal{J})}(t)}{h} - a_{\mathcal{P}}(t)y_{\mathcal{P}}^{(\mathcal{J})}(t) \Bigg| \qquad \text{almost everywhere.}$$

Hence, for almost all $t \in (0,1)$, $y_{\mathscr{P}}^{(\mathscr{I})\prime}(t) = a_{\mathscr{P}}(t) y_{\mathscr{P}}^{(\mathscr{I})}(t)$ and $y_{\mathscr{P}}^{(\mathscr{I})}(0) = x_{\mathscr{P}}^{(\mathscr{I})}(0)$. Furthermore, it is immediately clear from inequality (3.4) that $y(t)$ absolutely continuous on compact subsets of (0,1) implies the same for the coordinate function $y_{\mathscr{P}}^{(\mathscr{I})}(t)$. Therefore $y_{\mathscr{P}}^{(\mathscr{I})}(t) = \exp(\int_0^t a_{\mathscr{P}}(s)ds) x_{\mathscr{P}}^{(\mathscr{I})}(0)$ on $[0,1]$, and (3.7) follows.

Now consider the case where $y(0)$ equals the vector w from Lemma 3.2. To complete the proof, it will suffice to show that as $t \to 0$, $y(t)$ does not converge to w. Let $\{h_q\}_{q=1}^{\infty}$ be a sequence of numbers such that $\lim_{q\to\infty} h_q = 0$, $h_q \geqq 1/2^q$ and $y'(h_q) = A(h_q)y(h_q)$. Then, by Remarks 3.1 and Lemma 3.2, for each positive integer q there exist positive integers $M > N > q$ such that

$$\begin{aligned} |y(h_q) - w|_\Omega &= \Big| \Big(\cdots, \Big\{ \exp\Big(\int_0^{h_q} a_q(s)ds \Big) - 1 \Big\} w_q, \Big\{ \exp\Big(\int_0^{h_q} a_{q+1}(s)ds \Big) - 1 \Big\} w_{q+1}, \\ &\qquad \Big\{ \exp\Big(\int_0^{h_q} a_{q+2}(s)ds \Big) - 1 \Big\} w_{q+2}, \cdots \Big) \Big|_\Omega \\ &= |(\cdots, (e^{-1} - 1)w_q, (e^{-1} - 1)w_{q+1}, (e^{-1} - 1)w_{q+2} \cdots)|_\Omega \\ &\geqq (1 - e^{-1}) |(w_q, w_{q+1}, w_{q+2}, \cdots)|_\Omega \\ &\geqq (1 - e^{-1}) \Big\| \sum_{p=N}^{M} w_p \Big\|_\infty \\ &= \tfrac{1}{2}(1 - e^{-1}). \end{aligned}$$

■

It is possible to find a function $f: [0,1] \to X$ and an increasing function L such that the inequality in condition (iv) of Section 1 is satisfied for the operators $A(t)$ of Definitions 3.1. Let L equal the identity function and define $f(0) = 0$, and for $t > 0$ and in the support of a_p, let $f(t) = (2a_p(t)e_p, 0, 0, \cdots)$. Hence, if $t \in \operatorname{supp}(a_p)$ and $s \in \operatorname{supp}(a_q)$ with $p \neq q$, then for all $\lambda \geqq 0$ and $x \in \Omega$ we have

$$\begin{aligned} |J_\lambda(t)x - J_\lambda(s)x|_\Omega &= \lambda |J_\lambda(t)[A(t) - A(s)]J_\lambda(s)x|_\Omega \\ &\leqq \lambda |A(t)x - A(s)x|_\Omega \\ &\leqq \lambda \max\{|a_p(t)| \|x_p\|_\infty, |a_q(s)| \|x_q\|_\infty, \|a_p(t)x_p - a_q(s)x_q\|_\infty\} \\ &\leqq 2\lambda \max\{|a_p(t)|, |a_q(s)|\} |x|_\Omega \\ &= \lambda |f(t) - f(s)|_\Omega L(|x|_\Omega). \end{aligned}$$

The case $p = q$ is handled similarly.

Observe that $f(t)$ is continuous on the interval $[0,1]$ everywhere except at $t = 0$. Therefore, except for this one point of discontinuity, the family $\{A(t)\}_{0 \leqq t \leqq 1}$

of the counterexample would satisfy the Crandall–Pazy conditions (i) through (iv). But it is precisely this one discontinuity point which prevents the product integral (1.2) from converging. It is even true that f is not integrable on $[0,1]$. In fact, if f is *any* vector-valued function containing 0 in its domain and such that for some increasing function L:

$$|J_\lambda(t)x - J_\lambda(s)x|_\Omega \leqq \lambda\,|f(t)-f(s)|_\Omega L(|x|_\Omega) \tag{3.8}$$

for all $\lambda \geqq 0$, $x \in \Omega$, and almost all s and t in $[0,1]$, then f cannot be integrable. To see this, define $u^{(N)}$ to be the unit vector $(e_1, e_2, \cdots, e_N, 0, 0, \cdots)$ and let $\lambda_N = 1/2^{N+3}$ and $s = 0$. Then (3.8) implies for every positive integer N:

$$\frac{1}{L(1)}|J_{\lambda_N}(t)A(t)u^{(N)}|_\Omega + |f(0)|_\Omega \leqq |f(t)|_\Omega. \tag{3.9}$$

Now,

$$\begin{aligned}\int_{2^{-(N+1)}}^{1/2} |J_{\lambda_N}(t)A(t)u^{(N)}|_\Omega dt &= \sum_{p=1}^{N}\int_{2^{-(p+1)}}^{2^{-p}} |J_{\lambda_N}(t)A(t)u^{(N)}|_\Omega dt\\ &= \sum_{p=1}^{N}\int_{2^{-(p+1)}}^{2^{-p}} \frac{|a_p(t)|}{1-\lambda_N a_p(t)}dt \geqq \frac{2}{3}\sum_{p=1}^{N}\int_{2^{-(p+1)}}^{2^{-p}} |a_p(t)|\,dt = 2N/3,\end{aligned} \tag{3.10}$$

where we have used the bound: $1/(1-\lambda_N a_p(t)) \geqq \frac{2}{3}$ for all $1 \leqq p \leqq N$ and $0 \leqq t \leqq 1$. Combining (3.9) and (3.10) gives

$$\int_0^1 |f(t)|_\Omega dt \geqq \lim_{N\to\infty}\frac{1}{L(1)}\int_{2^{-(N+1)}}^{1/2} |J_{\lambda_N}A(t)u^{(N)}|_\Omega dt + |f(0)|_\Omega = +\infty. \tag{3.11}$$

Thus f is not integrable on $[0,1]$.

Evans [6], [7] gives conditions for the existence of the product integral which are the same as (i)–(iv) except that conditions (iii) and (iv) need only be true for almost every $s, t \in [0, T]$ and for some integrable f. (See remark 10.2 of [6].) In [6], [7] under these more general conditions, the product integral is shown to exist in the following sense. For every $t \in (0, T]$ there exists a sequence of partitions $\tau^{(n)} = (\tau_i^{(n)})_{i=1}^{N(n)}$ of $[0, t]$ with mesh sizes approaching zero such that for all $x_0 \in \bar{D}$,

$$\lim_{n\to\infty}\prod_{i=1}^{N(n)} J_{\tau_i^{(n)}-\tau_{i-1}^{(n)}}(\tau_i^{(n)})x_0$$

exists. As shown above, the operators $A(t)$ of the counterexample do not satisfy the Evans conditions. Thus it is no surprise that the product integral $\Pi_0^t J_{d\xi}(\xi)w$ of the counterexample does not exist in the Evans sense just described. This can be seen from the proof of Corollary 3.1.

The next section illustrates that even with $f(t)$ not integrable on $[0,1]$, the product integral may exist.

4. Example

The purpose of this section is to exhibit a family of operators $A(t)$ on a Banach space X which satisfies conditions (i), (ii) and (iii), but which does not satisfy the inequality in (iv) for any integrable vector-valued function f, and for which the product integral (1.2) exists for every $x \in X$. This will then serve to categorize the Crandall–Pazy and Evans conditions as sufficient but not necessary for the existence of the product integral. We first remark that if A is any operator-norm continuous map from $[0,T]$ into the bounded linear operators on a Banach space X, then while A may not satisfy condition (iv) for any vector-valued function f, it will satisfy (iv) for $f(t)$ operator-valued and equal to $A(t)$. In this case, (1.2) will exist as follows from elementary arguments (cf. [4, pp. 6, 80]) or as follows by the Crandall–Pazy convergence arguments in [1] which work equally well for f vector-valued or operator-valued.

Consider now $\{A(t)\}_{0\leqq t\leqq 1}$ equal to the family of diagonal operators given in Definitions 3.1, except that each $A(t)$ has domain $c_0 = \{\{x_p\}_{p=1}^{\infty} \in \ell_\infty : \lim_{p\to\infty} x_p = 0\}$, instead of the space Ω. Verification of properties (i)′, (ii)′ and (iii)′ of Section 1 is trivial and left to the reader. Furthermore, we claim that for each $x = \{x_p\}_{p=1}^{\infty} \in c_0$, the product integral (1.2) exists and equals $\{\exp(\int_0^t a_p(s)\,ds)x_p\}_{p=1}^{\infty}$. That is,

$$\lim_{q\to\infty}\left\| \prod_{i=1}^{q} [I-(t/q)A(it/q)]^{-1}x - \left\{\exp\left(\int_0^t a_p(s)\,dx\right)x_p\right\}_{p=1}^{\infty}\right\|_{\infty} \tag{4.1}$$
$$= \lim_{q\to\infty}\sup_{p\geqq 1}|g(p,q)x_p| = 0,$$

where, as in Proposition 3.1,

$$g(p,q) = \prod_{i=1}^{q}[1-(t/q)a_p(it/q)]^{-1} - \exp\left(\int_0^t a_p(s)\,ds\right).$$

The limit in (4.1) easily follows since: for each fixed p, $\lim_{q\to\infty} g(p,q) = 0$ (see [4, p. 52]); $|g(p,q)| \leqq 2$ for all p and q; and $x_p \to 0$ as $p\to\infty$. Finally, we argue that the inequality of condition (iv) cannot be satisfied for this example for any vector- or operator-valued function f which is integrable on $[0,1]$. This can be seen by letting $v^{(N)} = e_1 + e_2 + \cdots + e_N$, so that each $v^{(N)}$ is a unit vector in c_0. Next, in inequalities (3.8) through (3.11) substitute $v^{(N)}$ for $u^{(N)}$ and replace the Ω-norm with the ℓ_∞ norm. The argument then follows.

We mention that Crandall and Pazy in [1] provide an alternate condition in place of condition (iv). This was subsequently generalized by Evans [6], [7] as follows:

(v) There exist a measurable function $f:[0,T]\to X$ of bounded variation and an increasing function L such that for $0<\lambda<\Lambda$, almost all $0\leqq s\leqq t\leqq T$ and $x\in\bar{D}$:

$$\|J_\lambda(t)x-J_\lambda(s)x\|\leqq\lambda\,\|f(t)-f(s)\|\,L(\|x\|)\{1+\|A(s)J_\lambda(s)x\|\}.$$

It can be shown that every statement made in this paper with respect to condition (iv) remains valid if we replace (iv) by condition (v).

5. Questions about separability

The space Ω contains a copy of ℓ_∞ in each of its coordinate spaces; hence it is not separable. However, choosing coordinate spaces all equal to ℓ_∞ in Ω was done as a matter of simplicity. We could have defined a family of spaces

$$\Omega_p=\left\{(x_i)_{i=1}^\infty : x_i\in\ell_p,\lim_{i\to\infty}\|x_i\|_p=0\text{ and }|(x_i)_i|_\Omega<\infty\right\},\qquad 1\leqq p\leqq\infty.$$

Then all of the results and proofs of Section 3 would remain true if for any $p\in[1,\infty]$, the space Ω were replaced by Ω_p. Furthermore, even though Ω_p is a proper subspace of Ω for $p<\infty$, we have

Proposition 5.1. *Each space Ω_p, $1\leqq p\leqq\infty$, is nonseparable.*

Proof. It will suffice to find an uncountable family $\mathcal{F}$ of vectors in Ω_p such that $|u-v|_\Omega\geqq 1$ for all u, $v\in\mathcal{F}$. Given a real number $r\in[0,1]$, suppose that r has binary expansion $r=.r_1r_2\cdots$, where $r_n\in\{0,1\}$. Define $v^{(r)}$ as the vector in Ω_p with components

$$v_j^{(r)}=r_nw_j\qquad\text{for }2^{n-1}<j\leqq 2^n,\quad n=1,2,\cdots,$$

where the vector w_j is given by equation (3.6). Now let $\mathcal{F}=\{v^{(r)}:r\in[0,1]\}$. Then given u, $v\in\mathcal{F}$, for some index n we have

$$|u-v|_\Omega=|(\cdots,w_{2^{n-1}+1},w_{2^{n-1}+2},\cdots,w_{2^n},\cdots)|_\Omega\geqq\left\|\sum_{j=2^{n-1}+1}^{2^n}w_j\right\|_\infty=1.\qquad\blacksquare$$

As seen from the proof of Proposition 5.1, nonseparability of Ω_p and existence of the vector w in Lemma 3.2 are closely related results. As such, nonseparability is an essential ingredient in the counterexample of Section 3. We might well ask if it would be possible to construct a counterexample for Propositions 3.1 and 3.2 in a separable Banach space. Alternatively, we pose the

CONJECTURE. *Given A which satisfies conditions* (3.1), (3.2) *and* (3.3) *on an infinite dimensional space X with special properties, for example,* $X = L^2$, *the product integral* (1.2) *exists for every* $t \in [0,1]$ *and* $x \in X$.

It may also be of interest to investigate whether the product integral of the counterexample exists in a weak sense. More generally, we ask, if for any A which satisfies conditions (3.1), (3.2) and (3.3) on an arbitrary Banach space, does the product integral of resolvents converge weakly?

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